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NON-COMMUTATIVE DIFFERENTIAL GEOMETRY AND STANDARD MODEL

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Abstract

We incorporate Sogami's idea in the standard model into our previous formulation of non-commutative differential geometry by extending the action of the extra exterior derivative operator on spinors defined over the discrete space-time $M_4 \times Z_2$. The extension consists in making it possible to require that the operator become nilpotent when acting on the spinors. It is shown that the generalized field strength leads to the most general, gauge-invariant Yang-Mills-Higgs lagrangian even if the extra exterior derivative operator is not nilpotent, while the fermionic part remains intact. The proof is given for a single Higgs model. The method is applied to reformulate the standard model by putting left-handed fermion doublets on the upper sheet and right-handed fermion singlets on the lower sheet with generation mixing among quarks being taken into account. We also present a matrix calculus of the method without referring to the discrete space-time.

§1 Introduction

The standard model of elementary particles has passed all experimental checks so far and there is no doubt as to its validity up to energies explored by present accelerators. Nonetheless, it is annoyed with many fundamentally unknown parameters and, moreover, it essentially relies on the little-understood Higgs mechanism. It is a common belief in particle physicists that the standard model is a low energy effective theory of more fundamental theory. But, what is the more fundamental theory? There are many attempts to break the present deadlock in particle physics by invoking interesting physical motivations and/or mathematical apparatus.

Among them there is one making use of quite unfamiliar mathematics. It is Connes' gauge theory[1)] which aims at geometrizing Yang-Mills-Higgs broken gauge theory in terms of his non-commutative geometry (abbreviated NCG hereafter). The fact that the successful particle model is a broken gauge theory led Connes to envisage fine structure of the space-time, which allows to naturally introduce symmetry breaking into the gauge theory by tensoring Dirac operators on continuous and finite spaces. The simplest such spaces are M_4, our 4-dimensional minkowski space-time, and Z_2, two-points space. Connes' theory has since been expounded by various authors[2)−9)].

In this paper we introduce Connes' approach to the readers using new version of non-commutative differential geometry based on the underlying space-time $X = M_4 \times Z_2$. The old version was proposed in Ref.9) by modifying Sitarz' formalism[8)] and will be up-graded below by incorporating Sogami's clever idea[10)] in the standard model. The most prominent feature of the new version is to start from fermions in accord with the assumption[1)] that the underlying fields in NCG are the spinor fields. In this respect our formalism and Sitarz' one further developed by Ding et al.[8)] are similar but alike only in appearance. We shall point out the differences more concretely in the text. We hope the present version will help the readers to understand Connes' gauge theory more easily dispensing with abstract Connes' mathematics. It is worthwhile studying NCG in simpler setting.

The plan of this paper goes as follows. In the next section we introduce the new formulation of differential calculus on X. It is an up-grade version of that formulated in Ref.9). We shall consider in §3 gauge theory over X and prove that it is nothing but Yang-Mills-Higgs broken gauge theory with a single Higgs field. The method will be

applied to the standard model in §4. We essentially reproduce the results of Ref.12). We shall present a matrix version of the method in the chiral space in §5 without referring to the discrete space-time. The last section is devoted to discussions. Two Appendices are included containing some remarks on our algebraic manipulations.

§2 New Differential Calculus on Discrete Space-Time X

Let $\psi(x,y)$ be a spinor defined on the discrete space-time $X = M_4 \times Z_2$, where $x \in M_4$ and $y = \pm$ denote two elements of Z_2. (We regard Z_2 as two-points space but not as a discrete group.) With respect to the internal symmetry it is assumed to gauge-transform as

$$\psi(x,y) \rightarrow {}^g\psi(x,y) = \rho(g(x,y))\psi(x,y), \tag{1}$$

where $g(x,y)$ is a local gauge function belonging to the gauge group G_y and ρ indicates the fermion representation (rep) of the gauge group. Note that $G_+ \neq G_-$, in general. By convention we assume that left-handed fermions are placed on the upper sheet labeled by $y = +$, whilst right-handed fermions are put on the lower sheet labeled by $y = -$. Consequently, we write $\psi(x,+) = \psi_L(x)$ and $\psi(x,-) = \psi_R(x)$, where $\psi_L(x) = P_+\psi(x)$ and $\psi_R(x) = P_-\psi(x)$ for a Dirac spinor $\psi(x)$ with chiral projection operators $P_\pm = \frac{1}{2}(1 \pm \gamma_5)$ satisfying $P_+^2 = P_+, P_-^2 = P_-$ and $P_+P_- = P_-P_+ = 0$.

We next introduce the generalized exterior derivative operator $\mathbf{d} = d + d_\chi$ acting on $\psi(x,y)$, where d is the ordinary exterior derivative operator and d_χ turns out to describe symmetry breaking in the theory. In order to present detailed construction of the broken gauge theory in the next section it is convenient to make basic definitions in this section. The operator $\mathbf{d}$ is defined by

$$\begin{aligned}
&\mathbf{d}\psi(x,y) = d\psi(x,y) + d_\chi\psi(x,y),\\
&d\psi(x,y) = \partial_\mu\psi(x,y)d\hat{x}^\mu,\\
&d_\chi\psi(x,y) = [M(y)\psi(x,-y) + iC(y)\psi(x,y)]\chi, \quad M(-y) = M^\dagger(y),\\
&d^2\hat{x}^\mu = d_\chi d\hat{x}^\mu = d\chi = d_\chi\chi = 0, \quad \hat{x}^\mu: \text{ dimensionless coordinates.}
\end{aligned} \tag{2}$$

Here the basis of the "cotangent space" of X is denoted by $\{d\hat{x}^\mu, \chi\}$, which is dual to the basis $\{\partial_\mu, \partial_\chi\}$ of the "tangent space" of X with, for instance, $\chi(\partial_\chi) = 1$ (in

mass dimension one)[1]. We assume that $dd_\chi + d_\chi d = 0$, hence $\mathbf{d}^2 = d_\chi^2$ because $d^2 = 0$. The symbol χ was first introduced by Sitarz[8)] in relation to Connes' NCG[1)]. We continue to employ it although we drastically differ from Sitarz' formalism as emphasized in Ref.9). For instance, we assume $d_\chi \chi = 0$ in contrast with Sitarz' assumption $d_\chi \chi = 2\chi \wedge \chi \neq 0$ [2]. The action $d_\chi \psi(x,y)$ contains two matrix-valued functions $M(y)$ and $C(y)$, both of which are assumed to be x-independent. The case $C(y) = 0$ reduces to the previous definition (I-31)[3]. Hence $M(y)$ is identical with the previous one[9)], i.e., we assume a linear map $M(y) : \mathcal{H}_{-y} \to \mathcal{H}_y$, where $\mathcal{H}_{\pm y}$ denote Hilbert spaces of spinors $\psi(x, \pm y)$. We also assume a linear map $C(y) : \mathcal{H}_y \to \mathcal{H}_y$. A possible presence of the term $C(y)$ in the definition for $d_\chi \psi(x,y)$ is suggested from the 2×2 matrix formulation[12)] of Sogami's method[10)]. The precise role thereof in our present formulation will be clarified in the next section.

To distinguish linear maps from $\mathcal{H}_y$ to $\mathcal{H}_{\pm y}$ we introduce the concept of grading. We assign even grade to $C(y)$ and odd grade to $M(y)$. In general, linear maps depend on x, so that we have both even and odd functions $f(x,y)$. By definition we should consider only the products $f(x,y)\psi(x,y')$ with $y' = (-1)^{\partial f} y$, where $\partial f = 0$ for even function f and $\partial f = 1$ for odd function f [4]. Such products should be consistently calculable by the usual matrix multiplication rule. Furthermore Leibniz rule for the derivatives must also be applicable. As for the ordinary exterior derivative d there is no problem:

$$d(f(x,y)\psi(x,y')) = (df(x,y))\psi(x,y') + f(x,y)(d\psi(x,y')). \tag{3}$$

[1]The "derivation" ∂_χ kinematically "generates" the fermion mass. To see this let us consider free Dirac lagrangian in the following form. $\mathcal{L}_D^0(x) = i\bar\psi(x)\gamma^\mu\partial_\mu\psi(x) - \bar\psi(x)M\psi(x) = i\bar\psi_L(x)\gamma^\mu\partial_\mu\psi_L(x) + i\bar\psi_R(x)\gamma^\mu\partial_\mu\psi_R(x) - \bar\psi_L(x)M\psi_R(x) - \bar\psi_R(x)M\psi_L(x)$. Defining $\partial_\chi\psi_L(x) = M\psi_R(x) + ic_L\psi_L(x)$, $\partial_\chi\psi_R(x) = M\psi_L(x) + ic_R\psi_R(x)$ and putting $\psi(x,+) = \psi_L(x)$ and $\psi(x,-) = \psi_R(x)$, we get $\mathcal{L}_D^0(x) = \sum_{y=\pm}[i\bar\psi(x,y)\gamma^\mu\partial_\mu\psi(x,y) - \bar\psi(x,y)\partial_\chi\psi(x,y)]$, where we have made use of the relations $\bar\psi_L\psi_L = \bar\psi_R\psi_R = 0$. The operator d_χ acts on the spinor $\psi(x,y)$ as $d_\chi\psi(x,y) = (\partial_\chi\psi(x,y))\chi$, which takes the form of the third equation of Eq.(2) provided $M(+) = M(-) = M, C(+) = c_L$ and $C(-) = c_R$. It will be shown in the next section that gauge fields arise from the covariantization of ∂_μ and Higgs fields from that of ∂_χ.

[2]The authors in Ref.11) claim that the relation $d_\chi\chi = 2\chi\wedge\chi$ is one of important characteristics in NCG. On the contrary, we shall see below that the assumption $d_\chi\chi = 0$ is quite consistent with NCG. The point is that there exist more than one definitions of the action of d_χ on the 0-form (and the spinor) since it is no longer a differential but difference operator.

[3]We refer to Eq.(31) in Ref.9) as Eq.(I-31).

[4] The degree ∂f of the grade is defined up to mod 2.

On the other hand, the extra exterior derivative d_χ is assumed to satisfy the graded Leibniz rule

$$d_\chi(f(x,y)\psi(x,y')) = (d_\chi f(x,y))\psi(x,y') + (-1)^{\partial f} f(x,y)(d_\chi \psi(x,y')). \tag{4}$$

The reason why we should have extra factor $(-1)^{\partial f}$ in the above equation even for 0-form $f(x,y)$ will become clear in the next section. According to the definition (1) we easily find

$$\begin{aligned} d_\chi(f(x,y)\psi(x,y')) &= [M(y)f(x,-y)\psi(x,-y') + iC(y)f(x,y)\psi(x,y')]\chi \\ &= [d_\chi(f(x,y))\psi(x,y') \\ &+(-1)^{\partial f} f(x,y)\{M(y')\psi(x,-y') + iC(y')\psi(x,y')\}\chi]. \end{aligned} \tag{5}$$

Assuming that $\partial_\chi f$ has opposite grade to that of f [5], Eq.(5) yields

$$\chi\psi(x,y) = \psi(x,-y)\chi, \tag{6}$$

$$d_\chi f(x,y) = [M(y)f(x,-y) - (-1)^{\partial f} f(x,y)M(y')]\chi, \tag{7}$$

and

$$C(y)f(x,y) = (-1)^{\partial f} f(x,y)C(y'). \tag{8}$$

Equation (7) was previously[9)] proposed (see, Eq.(I-1)). By assumption (1) the gauge function $\rho(g(x,y))$ is even so that it commutes with $C(y)$ from Eq.(8):

$$C(y)\rho(g(x,y)) = \rho(g(x,y))C(y). \tag{9}$$

Taking $f(x,y) = M(y)$ in Eqs.(7) and (8) and remembering that $\partial M(y) = 1$ we have

$$\begin{aligned} d_\chi M(y) \quad &= 2M(y)M(-y)\chi \\ C(y)M(y) \quad &= -M(y)C(-y). \end{aligned} \tag{10}$$

As we showed in Ref.9), consistent calculability based on Eq.(7) implies that even functions $f(x,+)$ and $f(x,-)$ are square matrices of dimensions m and n, respectively,

[5] For any functions f, g we have $\partial(fg) = \partial f + \partial g$ mod 2.

while odd functions $f(x,+)$ and $f(x,-)$ are matrices of types (m,n) and (n,m), respectively. Consequently, $\psi(x,+)$ is m-component spinor and $\psi(x,-)$ n-component spinor regarding the internal symmetry groups G_+ and G_-, respectively. This presents a strong correlation between rep contents of fermions and bosons (gauge *and* Higgs fields). We also require the graded Leibniz rule for the product of linear maps $f(x,y)$ and $f'(x,y')$ with $y' = (-1)^{\partial f} y$

$$d_\chi(f(x,y)f'(x,y')) = (d_\chi f(x,y))f'(x,y') + (-1)^{\partial f} f(x,y)(d_\chi f'(x,y')), \tag{11}$$

which leads to Sitarz' relation like Eq.(6)

$$\chi f(x,y) = f(x,-y)\chi. \tag{12}$$

Conversely, if we assume Eq.(12), we can prove the graded Leibniz rule (11) from Eq.(7). The proof was given in the Appendix A of Ref.9). It is impossible to exaggerate that the relations (6) and (12) as they stand are not matrix equations. In particular, we consider[9)] an algebraic sum $A + B\chi$ of two matrices A and B of, in general, different types, as far as the variable y is explicit, while Sitarz[8)] assume A and B to be of the same type. Ding et al.[8)] assumed Eqs.(6) and (12) as matrix equations and ended up with the commutativity and anticommutativity of χ with bosonic and fermionic variables, respectively. In their formalism with Z_2 being taken as a symmetry group consisting of elements $\{e, r = (CPT)^2; r^2 = e\}$, therefore, χ is a commuting or anticommuting "scalar" in the matrix multiplication law.

If we employ the 2×2 matrix rep of elements of the Z_2-graded algebra[14)], we can combine Eqs.(6) and (12) into matrix equations by considering both $y = \pm$ cases simultaneously. More about this in the Appendix A.

According to Eq.(2) the operator d_χ is not diagonal in the sense that $d_\chi\psi(x,y)$ contains both even and odd functions $C(y)$ and $M(y)$. The operator d_χ^2 becomes diagonal, however, since the function $[M(y)M(-y) - C(y)C(y)]$ is even:

$$d_\chi^2\psi(x,y) = [M(y)M(-y) - C(y)C(y)]\psi(x,y)\chi\wedge\chi. \tag{13}$$

Similarly, we obtain[9)] from Eq.(7) that

$$d_\chi^2 f(x,y) = [M(y)M(-y)f(x,y) - f(x,y)M(y')M(-y')]\chi\wedge\chi,\ y' = (-1)^{\partial f} y. \tag{14}$$

This suggests that it is possible to assume the nilpotency of the operator d_χ by putting $M(y)M(-y) = C(y)C(y)$ [6]. This requirement precisely corresponds to the condition $M^2 = C^2$ as observed in Ref.13) where it was shown that Sogami's method[10)] is equivalent to one version[14)] of NCG. In the present formulation we shall not need the nilpotency of d_χ and make use of Eq.(13) in the next section even if $M(y)M(-y) \neq C(y)C(y)$. However, it is interesting to remark that the present formalism is applicable to both nilpotent and non-nilpotent cases. In this respect, too, we are differing from Ref.8) where the nilpotency of d_χ is the central requirement.

§3 Gauge Theory on Discrete Space-Time X

Although the present formalism is applicable to both global and local symmetries, we consider exclusively local gauge theories in this article. It is apparent that $\mathbf{d}\psi(x,y)$ is not covariant under local gauge transformations (1). It is covariantized by the familiar recipe:

$$\mathcal{D}(x,y)\psi(x,y) = (\mathbf{d} + (\rho_*\mathbf{A})(x,y))\psi(x,y), \tag{15}$$

where ρ_* is the differential rep for the fermions and the generalized gauge potential

$$\mathbf{A}(x,y) = A(x,y) + \Phi(x,y)\chi, \quad A(x,y) = A_\mu(x,y)d\hat{x}^\mu, \tag{16}$$

is subject to the inhomogeneous gauge transformation

$${}^g\mathbf{A}(x,y) = g(x,y)\mathbf{A}(x,y)g^{-1}(x,y) + g(x,y)\mathbf{d}g^{-1}(x,y). \tag{17}$$

The notation ρ_* means $(\rho_*\mathbf{A})(x,y) = (\rho_*A)(x,y) + \Phi(x,y)\chi$, which transforms like Eq.(17) with $g(x,y) \to \rho(g(x,y))$. In particular, we have

$${}^g\Phi(x,y) = g(x,y)\Phi(x,y)g^{-1}(x,-y) + g(x,y)\partial_\chi g^{-1}(x,-y). \tag{18}$$

[6] It is interesting to emphasize that it is possible to realize the nilpotency of d_χ without Sitarz' assumption $d_\chi\chi = 2\chi\wedge\chi$. This observation suggests itself that realization of Connes' NCG in terms of the new symbol χ is not unique. In other words, there exist various definitions of d_χ though d_χ is uniquely determined on the whole algebra once its action on the 0-form (or the spinor) is defined. See the second footnote on p.4.

This inhomogeneous transformation law, valid also for $\rho(g(x,y))$ in place of $g(x,y)$, will be rederived in the Appendix B in a different way and is entirely different from that proposed in Ref.8). It stems from the different definitions of the operator ∂_χ. Our $\Phi(x,y)$ is a genuine *shifted* Higgs field, but the scalar field in the generalized one-form in Ref.8) represents unshifted Higgs field, although the scalar field itself transforms inhomogeneously unless gauge transformations defined at e and r in the notation of Ding et al.[8] are identical to each other. It follows that, if we define the *back-shifted*[3] Higgs field

$$H(x,y) = \Phi(x,y) + M(y), \tag{19}$$

Eq.(18) implies the homogeneous transformation law for it:

$${}^{g}H(x,y) = g(x,y)H(x,y)g^{-1}(x,-y). \tag{20}$$

Hence, $M(y)$ determines VEV of the Higgs field $< H(x,y) >= M(y)$. The gauge group $G_+ \times G_-$ is broken down to $H_+ \times H_-$, where $d_\chi h^{-1}(x,y) = 0$ for $h(x,y) \in H_y$. The matrix $M(y)$ determines[9] the scale and the pattern of spontaneous symmetry breaking. Higgs field enters into the theory under the different guise in Sitarz' formalism.

By construction $\mathcal{D}(x,y)\psi(x,y)$ is gauge-covariant under Eq.(1) and can be rewritten as

$$\begin{aligned} \mathcal{D}(x,y)\psi(x,y) &= [D(x,y) + d_\chi + \Phi(x,y)\chi]\psi(x,y) \\ &= D(x,y)\psi(x,y) + [H(x,y)\psi(x,-y) + iC(y)\psi(x,y)]\chi, \end{aligned} \tag{21}$$

where

$$D(x,y) = d + (\rho_* A)(x,y). \tag{22}$$

The presence of the extra one-form basis χ prevents us from representing the covariant derivative (15) or (21) in terms of Clifford algebra. Instead we introduce the associated spinor one-form

$$\tilde{\psi}(x,y) = \gamma_\mu \psi(x,y) d\hat{x}^\mu - i\alpha^{-2}\psi(x,y)\chi \tag{23}$$

for some constant α^{-2}. Dirac γ-matrices are taken to satisfy $\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=2g_{\mu\nu}$ with ${\gamma_\mu}^\dagger=\gamma_0\gamma_\mu\gamma_0$ and $\gamma_5=-i\gamma_0\gamma_1\gamma_2\gamma_3$. The Dirac lagrangian is then computed by the inner product

$$\begin{aligned}\mathcal{L}_D(x,y) &= i<\tilde{\psi}(x,y),\mathcal{D}(x,y)\psi(x,y)> \\ &= \bar{\psi}(x,y)i\gamma^\mu D_\mu(x,y)\psi(x,y)-\bar{\psi}(x,y)H(x,y)\psi(x,-y) \\ &\quad -i\bar{\psi}(x,y)C(y)\psi(x,y),\end{aligned} \tag{24}$$

where we have defined the inner products of spinor one-forms through

$$\begin{aligned}&<\psi(x,y)d\hat{x}^\mu,\psi'(x,y)d\hat{x}^\nu>=\bar{\psi}(x,y)\psi'(x,y)g^{\mu\nu} \\ &<\psi(x,y)\chi,\psi'(x,y)\chi>=\bar{\psi}(x,y)\psi'(x,y)<\chi,\chi>\end{aligned} \tag{25}$$

with vanishing other inner products, and we have put

$$<\chi,\chi>=-\alpha^2. \tag{26}$$

Here the overbar of the spinor dictates the Pauli adjoint: $\bar{\psi}(x,y)=\psi^\dagger(x,y)\gamma^0$. By our assumption of the assignment of spinors $\bar{\psi}(x,y)C(y)\psi(x,y)=0$ for $y=\pm$ because of the relations $P_+P_-=P_-P_+=0$. Thus $C(y)$ disappears from the Dirac lagrangian (24) and is nothing but Sogami's term[10)], called $c_{L,R}$ in Ref.12). The total Dirac lagrangian is the sum over $y=\pm$:

$$\begin{aligned}\mathcal{L}_D(x) &= \textstyle\sum_{y=\pm}\mathcal{L}_D(x,y) \\ &= \textstyle\sum_{y=\pm}[\bar{\psi}(x,y)i\gamma^\mu D_\mu(x,y)\psi(x,y)-\bar{\psi}(x,y)H(x,y)\psi(x,-y)],\end{aligned} \tag{27}$$

which is hermitian since $H(x,-y)=H^\dagger(x,y)$.

Up to now we have concentrated on the fermionic sector. The bosonic sector is characterized by the generalized field strength

$$\begin{aligned}\mathcal{F}(x,y) &= [\mathbf{d}+(\rho_*\mathbf{A})(x,y)]\wedge[\mathbf{d}+(\rho_*\mathbf{A})(x,y)] \\ &= \mathcal{D}(x,y)\wedge\mathcal{D}(x,y) \\ &= [D(x,y)+d_\chi+\Phi(x,y)\chi]\wedge[D(x,y)+d_\chi+\Phi(x,y)\chi].\end{aligned} \tag{28}$$

Since $\mathbf{d}$ is not necessarily nilpotent, it differs from

$$\mathbf{F}(x,y)=\mathbf{d}(\rho_*\mathbf{A})(x,y)+(\rho_*\mathbf{A})(x,y)\wedge(\rho_*\mathbf{A})(x,y), \tag{29}$$

which is not gauge covariant unless $\mathbf{d}^2 = 0$. To evaluate Eq.(28) we need the expressions for $d_\chi(\rho_* A)_\mu(x,y)$ and $d_\chi \Phi(x,y)$ [7]. We determine them by requiring the graded Leibniz rule for the generalized one-forms (16):

$$\mathbf{d}(\mathbf{A}(x,y) \wedge \mathbf{B}(x,y)) = (\mathbf{dA}(x,y)) \wedge \mathbf{B}(x,y) - \mathbf{A}(x,y) \wedge (\mathbf{dB}(x,y)). \tag{30}$$

This is necessary to show the gauge covariance of Eq.(28). We proved in the Appendix A of Ref.9) that Eq.(30) is valid provided that the ordinary gauge field $A_\mu(x,y)$ is even, while the shifted Higgs field $\Phi(x,y)$ is odd, which is consistent with the assumption $\partial M(y) = 1$ in view of Eq.(19). The factor $(-1)^{\partial f}$ in Eq.(11) is essential to this proof. This therefore fixes the operation d_χ on $(\rho_* A)_\mu(x,y)$ and $\Phi(x,y)$ according to Eq.(7). Substituting $d_\chi^2 = [M(y)M(-y) - C(y)C(y)]\chi \wedge \chi$ from Eq.(13) into Eq.(28) we finally find that

$$\mathcal{F}(x,y) = F(x,y) + DH(x,y) \wedge \chi + (H(x,y)H(x,-y) - C(y)C(y))\chi \wedge \chi, \tag{31}$$

where we put

$$F(x,y) = D(x,y) \wedge D(x,y) = \frac{1}{2} F_{\mu\nu}(x,y) d\hat{x}^\mu \wedge \hat{x}^\nu, \tag{32}$$

and assume $d\hat{x}^\mu \wedge \chi = -\chi \wedge d\hat{x}^\mu$ to obtain

$$DH(x,y) = D(x,y)H(x,y) - H(x,y)D(x,-y) = (D_\mu H(x,y)) d\hat{x}^\mu. \tag{33}$$

The bosonic lagrangian is then given by the sum[8]

$$\begin{array}{ll} \mathcal{L}_B(x) & = \sum_{y=\pm} \mathcal{L}_B(x,y), \\ \mathcal{L}_B(x,y) & = -\mathrm{tr}\frac{1}{4g_y^2} < \mathcal{F}(x,y), \mathcal{F}(x,y) >, \end{array} \tag{34}$$

[7] In addition to the usual differential calculus we assume $d \wedge \chi + \chi \wedge d = 0$, $d_\chi \wedge d\hat{x}^\mu + d\hat{x}^\mu \wedge d_\chi = 0$, $d_\chi \wedge \chi - \chi \wedge d_\chi = 0$. The last relation implies $d_\chi \wedge (\chi\psi) = (d_\chi \chi)\psi + \chi \wedge d_\chi \psi = \chi \wedge d_\chi \psi$ because $(d_\chi \chi) = 0$ by assumption.

[8] The fact that the covariant derivative $\mathcal{D}(x,y)$ contains both gauge and Higgs fields which couple to fermions, gives sever restriction on the form of the bosonic lagrangian (34), although the number of parameters in the bare lagrangian should be the same as that for renormalizable theory unless some hidden symmetry is present.

where $\frac{1}{4g_y^2}$ is a coupling-constants-matrix commuting with the gauge transformation $\rho(g(x,y))$ and tr indicates the trace over the internal symmetry matrices. Here the inner products are to be evaluated through

$$
\begin{aligned}
&< d\hat{x}^\mu \wedge d\hat{x}^\nu, d\hat{x}^\rho \wedge d\hat{x}^\sigma >= g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho} \\
&< d\hat{x}^\mu \wedge \chi, d\hat{x}^\nu \wedge \chi >= g^{\mu\nu}(-\alpha^2) \\
&< \chi \wedge \chi, \chi \wedge \chi >= \tfrac{\beta^2}{2},
\end{aligned}
\tag{35}
$$

while other inner products of basis two-forms are vanishing.

To summarize we find the formula

$$
\begin{aligned}
\mathcal{L}_B(x) = \sum_{y=\pm}[&-\frac{1}{2}\mathrm{tr}\frac{1}{4g_y^2}F_{\mu\nu}(x,y)F^{\mu\nu}(x,y) + \alpha^2\mathrm{tr}\frac{1}{4g_y^2}(D_\mu H(x,y))^\dagger D^\mu H(x,y) \\
&-\tfrac{1}{2}\beta^2\mathrm{tr}\tfrac{1}{4g_y^2}(H(x,y)H(x,-y) - C(y)C(y))^2].
\end{aligned}
\tag{36}
$$

Since $C(y)$ is gauge invariant because of Eq.(9), the bosonic lagrangian (36) is the most general, gauge invariant Yang-Mills-Higgs lagrangian provided there exists only one Higgs field. The total lagrangian, the sum of Eqs.(27) and (36), becomes identical with that obtained in the second reference of Ref.8) provided $C(+) = C(-)$ and $\beta^2 = 2\alpha^4$. Consequently, it is not necessary to regard Z_2 as a discrete group composed of $\{1, (CPT)^2\}$. This group structure enforces the identification $\psi(x,y) = -\psi(x,-y)$ which contradicts with our assignment $\psi(x,+) = \psi_L(x)$ and $\psi(x,-) = \psi_R(x)$. The latter is more close to Connes' assignment[1)].

The result (36) is markedly different from our previous ones (I-19) and (I-11) which contain not-necessarily gauge-invariant term $M(y)M(-y)$ in the Higgs potential. The latter disappears from the scene by the introduction of the term $C(y)$ in the definition for $d_\chi\psi(x,y)$. Thus in the present formalism we are not forced to discard gauge-noninvariant term $\mathrm{tr}(H(x,+)H(x,-) - M(+)M(-))^2$ of Model I in Ref.9) by hand. It is simply replaced with gauge-invariant one $\mathrm{tr}(H(x,+)H(x,-) - C(+)C(+))^2$. Nevertheless, the minimum of the Higgs potential should occur at $H(x,y) = M(y)$.

§4 Application to Standard Model

In this section we apply the previous formalism to reformulate the standard model taking the generation mixing among quarks into account.

To this end let us first recall that the Dirac lagrangian for the standard model is given by the sum of leptonic and quark parts

$$\mathcal{L}_D = \mathcal{L}_D^{(l)} + \mathcal{L}_D^{(q)}. \tag{37}$$

To fix the notation we recapitulate it. Writing the weak lepton doublets and singlets, respectively, in the i-th generation, as $l_L^i = \begin{pmatrix} \nu_{eL}^i \\ e_L^i \end{pmatrix}$ and e_R^i, the leptonic part reads

$$\begin{aligned} \mathcal{L}_D^{(l)} = \; & \textstyle\sum_i \bar{l}_L^i i\gamma^\mu(\partial_\mu - i\frac{g}{2}\vec{\tau}\cdot\vec{A}_\mu + i\frac{g'}{2}B_\mu)l_L^i \\ & + \textstyle\sum_i \bar{e}_R^i i\gamma^\mu(\partial_\mu + ig'B_\mu)e_R^i - \sum_{i,j}[a_{ij}^{(e)}\bar{l}_L^i\phi e_R^j + a_{ji}^{(e)*}\bar{e}_R^i\phi^\dagger l_L^j], \end{aligned} \tag{38}$$

where $\vec{A}_\mu$ and B_μ are $SU(2)$ and $U(1)$ gauge fields, respectively, with corresponding gauge coupling constants g and g', $\tau_i (i = 1, 2, 3)$ Pauli matrices, ϕ stands for Higgs doublet $\begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$ and $a_{ij}^{(e)}$ represent the Yukawa coupling constants. It is possible to diagonalize the matrix $a^{(e)} = (a_{ij}^{(e)})$ in the generation space without changing gauge interactions of leptons as far as neutrinos are assumed to be massless. On the other hand, the quark sector is defined by the lagrangian

$$\begin{aligned} \mathcal{L}_D^{(q)} = \; & \textstyle\sum_{\alpha,\beta,i} \bar{q}_L^{\alpha i} i\gamma^\mu(\partial_\mu\delta_{\alpha\beta} - i\frac{g_s}{2}(\lambda^a)_{\alpha\beta}G_\mu^a - i\frac{g}{2}\vec{\tau}\cdot\vec{A}_\mu\delta_{\alpha\beta} - i\frac{g'}{6}B_\mu\delta_{\alpha\beta})q_L^{\beta i} \\ & + \textstyle\sum_{\alpha,\beta,i} \bar{d}_R^{\alpha i} i\gamma^\mu(\partial_\mu\delta_{\alpha\beta} - i\frac{g_s}{2}(\lambda^a)_{\alpha\beta}G_\mu^a + i\frac{g'}{3}B_\mu\delta_{\alpha\beta})d_R^{\beta i} \\ & + \textstyle\sum_{\alpha,\beta,i} \bar{u}_R^{\alpha i} i\gamma^\mu(\partial_\mu\delta_{\alpha\beta} - i\frac{g_s}{2}(\lambda^a)_{\alpha\beta}G_\mu^a - i\frac{2g'}{3}B_\mu\delta_{\alpha\beta})u_R^{\beta i} \\ & - \textstyle\sum_{\alpha,i,j}[a_{ij}^{(d)}(\bar{q}_L^{\alpha i}\phi)d_R^{\alpha j} + a_{ji}^{(d)*}\bar{d}_R^{\alpha i}(\phi^\dagger q_L^{\alpha j})] \\ & - \textstyle\sum_{\alpha,i,j}[a_{ij}^{(u)}(\bar{q}_L^{\alpha i}\tilde{\phi})u_R^{\alpha j} + a_{ji}^{(u)*}\bar{u}_R^{\alpha i}(\tilde{\phi}^\dagger q_L^{\alpha j})], \end{aligned} \tag{39}$$

where $\tilde{\phi} = i\tau_2\phi$, left-handed quark doublet and right-handed quark singlets in i-th generation with color α are designated by $q_L^{\alpha i} = \begin{pmatrix} u_L^{\alpha i} \\ d_L^{\alpha i} \end{pmatrix}$, and $u_R^{\alpha i}$ and $d_R^{\alpha i}$, respectively, $G_\mu^a (a = 1, 2, \cdots, 8)$ stands for the gluon fields, $\lambda^a (a = 1, 2, \cdots, 8)$ are the Gell-Mann matrices, g_s denotes the QCD coupling constant and $a_{ij}^{(q)}, q = u, d$, represent Yukawa coupling matrices. We assume N_g generations, $i, j = 1, 2, \cdots, N_g$. It is possible to diagonalize the matrices $a^{(q)} = (a_{ij}^{(q)})$ by biunitary transformations, i.e., $U^{(q)}a^{(q)}V^{(q)\dagger} = g^{(q)}$ are chosen to be diagonal matrices with real, positive eigenvalues for some unitary matrices $U^{(q)}$ and $V^{(q)}$. Then, the gauge interactions of quarks are written in terms of mass eigenstates, where Kobayashi-Maskawa matrix

$U = U^{(u)}U^{(d)-1}$ appears for the charged current interactions. In what follows we prefer to use gauge eigenstates as exhibited in Eq.(39). The non-diagonal matrices $a^{(q)}, q = u, d$ indicate the generation mixing among quarks.

Sogami[10)] proposed to derive the bosonic lagrangian from the sum of Eqs.(38) and (39) and obtained a constrained standard model[9]. Subsequently, the constraints are removed by noting[12)] that Sogami's method[10)] allows more parameters than originally supposed. Then, the bosonic lagrangian is also given by the sum

$$\mathcal{L}_B = \mathcal{L}_B^{(l)} + \mathcal{L}_B^{(q)}. \tag{40}$$

The relative weight between the leptonic and quark contributions in Eq.(40) is determined only phenomenologically in the tree level.

In the present formalism where $\psi(x,+) = \psi_L(x)$ and $\psi(x,-) = \psi_R(x)$ we should place the left-handed leptons *and* quarks on the upper sheet, while the right-handed leptons *and* quarks are to be put on the lower sheet. This needs a reconsideration of the derivation of Eq.(40).

It is not difficult to cast the sum of Eqs.(38) and (39) into the form (27) by choosing

$$\begin{aligned} \psi^i(x,+) &= \begin{pmatrix} l_L^i(x) \\ q_L^{\alpha i}(x) \end{pmatrix} \\ \psi^i(x,-) &= \begin{pmatrix} e_R^i(x) \\ d_R^{\alpha i}(x) \\ u_R^{\alpha i}(x) \end{pmatrix}. \end{aligned} \tag{41}$$

In what follows we shall omit the generation and color indices. Since, in each generation, the left-handed fermions are flavor doublets, the right-handed fermions singlets and quarks exist in three colors, $\alpha = R, B, G$, $\psi(x,+)$ is 8-component spinor regarding the gauge group $G_+ = U(2) \otimes SU(3)$, and $\psi(x,-)$ consists of 7 components for the gauge group $U(1) \otimes SU(3)$. Consequently, the ordinary covariant derivatives $D(x,+) = d + \mathcal{A}(x,+)$ and $D(x,-) = d + \mathcal{A}(x,-)$ are 8×8 and 7×7 matrices, respectively, while $H(x,+) = \Phi(x,+) + M(+)$ is 8×7 matrix and $H(x,-) = H^\dagger(x,+)$ 7×8 matrix. This is valid for every generation. In addition we should consider the

[9]Sogami's reconstruction of the standard model lagrangian in the bosonic sector is quite different from that of Connes' NCG although the constraints are more or less similar. Our aim is to greatly simplify Connes' NCG in relation to Sogami's method.

generation space matrix, either unit matrix or Yukawa coupling matrices as direct products which are to be understood in the following expressions.

Denoting p-dimensional unit matrix by 1_p and looking at Eqs.(38) and (39), the assignment (41) gives

$$\begin{aligned}\mathcal{A}(x,+) &= -\tfrac{ig}{2}A(x)\otimes 1_4 - \tfrac{ig'}{2}B(x)Y_L + 1_2\otimes(-\tfrac{ig_s}{2}G(x))\\ \mathcal{A}(x,-) &= -\tfrac{ig'}{2}B(x)Y_R - \tfrac{ig_s}{2}\tilde{G}(x)\end{aligned} \tag{42}$$

and

$$H(x,+) = \begin{pmatrix} a^{(e)}\phi(x) & 0 & 0 \\ 0 & a^{(d)}\phi(x)\otimes 1_3 & a^{(u)}\tilde{\phi}(x)\otimes 1_3 \end{pmatrix}, \tag{43}$$

where $A(x) = \tau_i A^i_\mu(x)d\hat{x}^\mu$, $B(x) = B_\mu(x)d\hat{x}^\mu$, $G(x) = \begin{pmatrix} 0 & 0 \\ 0 & \lambda_a G^a_\mu(x)d\hat{x}^\mu \end{pmatrix}$ and $\tilde{G}(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda_a G^a_\mu(x)d\hat{x}^\mu & 0 \\ 0 & 0 & \lambda_a G^a_\mu(x)d\hat{x}^\mu \end{pmatrix}$. The hypercharge matrices for fermions are denoted by

$$\begin{aligned}Y_L &= \begin{pmatrix} -1\cdot 1_2 & 0 \\ 0 & \tfrac{1}{3}\cdot 1_2\otimes 1_3 \end{pmatrix}\\ Y_R &= \begin{pmatrix} -2 & 0 & 0 \\ 0 & -\tfrac{2}{3}\cdot 1_3 & 0 \\ 0 & 0 & \tfrac{4}{3}\cdot 1_3 \end{pmatrix}.\end{aligned} \tag{44}$$

Our next task is to evaluate the generalized field strength $\mathcal{F}(x,y)$ of Eq.(31). The ordinary field strength $F(x,y)$ takes the form

$$\begin{aligned}F(x,+) &= -\tfrac{ig}{2}f(x)\otimes 1_4 - \tfrac{ig'}{2}f^0(x)Y_L + 1_2\otimes(-\tfrac{ig_s}{2}G^{(L)}(x))\\ F(x,-) &= -\tfrac{ig'}{2}f^0(x)Y_R - \tfrac{ig_s}{2}G^{(R)}(x),\end{aligned} \tag{45}$$

where

$$\begin{aligned}f(x) &= \tfrac{1}{2}\vec{\tau}\cdot\vec{f}_{\mu\nu}d\hat{x}^\mu\wedge d\hat{x}^\nu,\\ f^0(x) &= \tfrac{1}{2}f^0_{\mu\nu}d\hat{x}^\mu\wedge d\hat{x}^\nu,\\ G^{(L)}(x) &= \tfrac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & \lambda_a G^a_{\mu\nu}(x)d\hat{x}^\mu\wedge d\hat{x}^\nu \end{pmatrix}\\ G^{(R)}(x) &= \tfrac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda_a G^a_{\mu\nu}(x)d\hat{x}^\mu\wedge d\hat{x}^\nu & 0 \\ 0 & 0 & \lambda_a G^a_{\mu\nu}(x)d\hat{x}^\mu\wedge d\hat{x}^\nu \end{pmatrix}\end{aligned} \tag{46}$$

with the usual field strengths

$$
\begin{aligned}
\vec{f}_{\mu\nu} &= \partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu + g\vec{A}_\mu \times \vec{A}_\nu, \quad f^0_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu, \\
G^a_{\mu\nu} &= \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g_s f_{abc} G_\mu{}^b G_\nu{}^c,
\end{aligned}
\tag{47}
$$

f_{abc} being $SU(3)$ structure constants.

The covariant derivative, $D_\mu H(x,y)$, of the Higgs field (43) is given by applying D_μ on ϕ and $\tilde{\phi}$ in Eq.(43) with

$$
\begin{aligned}
D_\mu \phi &= (\partial_\mu - i\tfrac{g}{2}\vec{\tau}\cdot\vec{A}_\mu - i\tfrac{g'}{2}B_\mu)\phi \\
D_\mu \tilde{\phi} &= (\partial_\mu - i\tfrac{g}{2}\vec{\tau}\cdot\vec{A}_\mu + i\tfrac{g'}{2}B_\mu)\tilde{\phi}.
\end{aligned}
\tag{48}
$$

Now we determine the form of the matrix $C(y)$ from the gauge invariance (9). It turns out that

$$
\begin{aligned}
C(+) &= \begin{pmatrix} c_L^{(l)} 1_2 & 0 \\ 0 & c_L^{(q)} 1_2 \otimes 1_3 \end{pmatrix} \\
C(-) &= \begin{pmatrix} c_R^{(e)} & 0 & 0 \\ 0 & c_R^{(d)} 1_3 & 0 \\ 0 & 0 & c_R^{(u)} 1_3 \end{pmatrix},
\end{aligned}
\tag{49}
$$

where $c_{L,R}^{(f)}, f = l, q, e, d, u$ are the constant matrices in the generation space. The condition $C(y)M(y) = -M(y)C(-y)$ of Eq.(10) is satisfied by $c_L^{(l)} a^{(e)} = -a^{(e)} c_R^{(e)}$, $c_L^{(q)} a^{(d)} = -a^{(d)} c_R^{(d)}$ and $c_L^{(q)} a^{(u)} = -a^{(u)} c_R^{(u)}$. The Higgs potential comes from the last term on the right-hand side of Eq.(31).

Finally we have to parametrize the coupling-constants-matrix $\frac{1}{4g_y^2} \equiv \frac{1}{4G^2} C_y^2$ in Eq.(34):

$$
\begin{aligned}
C_+^2 &= \begin{pmatrix} 1_2 & 0 \\ 0 & C_Q^2 1_2 \otimes 1_3 \end{pmatrix} \\
C_-^2 &= \begin{pmatrix} \delta^2 & 0 & 0 \\ 0 & \delta^2 C_Q^2 1_3 & 0 \\ 0 & 0 & \delta^2 C_Q^2 1_3 \end{pmatrix},
\end{aligned}
\tag{50}
$$

where we have introduced two more parameters δ^2 and C_Q^2. The most general form of C_-^2 contains different parameters C_d^2 and C_u^2 which are assumed to be equal to C_Q^2

in Eq.(50). By choosing

$$
\begin{aligned}
G^2 &= \tfrac{g_s^2}{2}N_g(1+\delta^2)C_Q^2,\\
G^2 &= \tfrac{g^2}{4}N_g(1+3C_Q^2),\\
G^2 &= \tfrac{g'^2}{8}N_g\mathrm{tr}[C_+^2Y_L^2+C_-^2Y_R^2] = \tfrac{g'^2}{4}N_g[(1+2\delta^2)+\tfrac{1}{3}C_Q^2(1+10\delta^2)],\\
G^2 &= \tfrac{\alpha^2}{4}(1+\delta^2)[\mathrm{tr}(a^{(e)\dagger}a^{(e)})+3C_Q^2\mathrm{tr}(a^{(d)\dagger}a^{(d)})+3C_Q^2\mathrm{tr}(a^{(u)\dagger}a^{(u)})],
\end{aligned}
\tag{51}
$$

we finally find that

$$
\begin{aligned}
\mathcal{L}_B &= -\textstyle\sum_{y=\pm}\mathrm{tr}\tfrac{1}{4g_y^2} < \mathcal{F}(x,y),\mathcal{F}(x,y) >\\
&= -\tfrac{1}{4}[G^a_{\mu\nu}G^{a,\mu\nu}+\vec{f}_{\mu\nu}\cdot\vec{f}^{\mu\nu}+f^0_{\mu\nu}f^{0,\mu\nu}]\\
&\quad +(D_\mu\phi)^\dagger(D^\mu\phi)-\tfrac{\lambda}{4}(\phi^\dagger\phi)^2+\mu^2(\phi^\dagger\phi)+\mathrm{const.},
\end{aligned}
\tag{52}
$$

where we have

$$
\lambda = 2\epsilon^2\frac{\mathrm{tr}(a^{(e)}a^{(e)\dagger})^2+3C_Q^2\mathrm{tr}(a^{(d)}a^{(d)\dagger})^2+3C_Q^2\mathrm{tr}(a^{(u)}a^{(u)\dagger})^2}{\mathrm{tr}(a^{(e)}a^{(e)\dagger})+3C_Q^2\mathrm{tr}(a^{(d)}a^{(d)\dagger})+3C_Q^2\mathrm{tr}(a^{(u)}a^{(u)\dagger})},
\tag{53}
$$

with $\epsilon^2 = \frac{\beta^2}{\alpha^2}$ [10] and, assuming $c_L^{(l)} = \varsigma, c_R^{(e)} = -\varsigma, c_L^{(q)} = \varsigma, c_R^{(d)} = -\varsigma$ and $c_R^{(d)} = -\varsigma$ for constant ς,

$$
\mu^2 = \epsilon^2\varsigma^2.
\tag{54}
$$

Equation (52) is nothing but the bosonic lagrangian in the standard model with the following parametrization for gauge coupling constants

$$
\frac{g_s^2}{g^2} = \frac{1+3C_Q^2}{2C_Q^2(1+\delta^2)}
\tag{55}
$$

and

$$
\frac{g'^2}{g^2} = \tan^2\theta_W = \frac{1+3C_Q^2}{(1+2\delta^2)+\frac{1}{3}C_Q^2(1+10\delta^2)}
\tag{56}
$$

where θ_W is Weinberg angle. Sogami[10)] emphasized that it is not necessary to imagine the two-sheeted world as far as the derivation of the bosonic lagrangian (52) is

[10] Sogami's λ is four times ours and obtained by putting $\epsilon^2 = 1$ and $C_Q^2 = 1$.

concerned. In fact, we shall see in the next section that the same lagrangian (52) is obtained by the matrix method without explicitly referring to the discrete space-time. We, therefore, conclude that our algebraic rule based on the algebra of functions over the discrete space-time, $X = M_4 \times Z_2$, defines only a convenient mathematical manipulation consistent with the chiral nature of fermions but the standard model itself is reconstructed solely on the continuous manifold M_4. Equations (55) and (56) were already derived in Ref.12) where additional parameter c appeared due to an alternative choice of the chiral spinors, leading to different equation for λ than Eq.(53).

If quarks and leptons contribute equally to the bosonic lagrangian, $C_Q^2 = 1$, and, moreover, δ^2 is taken to be unity, we obtain $SU(5)$ relations $g_s^2 = g^2 = \frac{5}{3}g'^2$. This is realized provided that C_+^2 and C_-^2 are unit matrices of dimensions 8 and 7, respectively, which, therefore, should reflect $SU(5)$ symmetry in some sense. As noted in Ref.12), rough estimation for $\frac{g_s^2}{g^2} \sim 4$ and $\frac{g'^2}{g^2} \sim \frac{1}{3}$ at present energy gives $C_Q^2 \sim \frac{1}{14}$ and $\delta^2 \sim \frac{7}{6}$. If we retain only top quark contribution to the quartic coupling constant (52) [11], we would have $m_H \approx \epsilon\sqrt{2}m_t$, where m_H and m_t denote physical Higgs boson and top quark masses, respectively. This is to be compared with Sogami's prediction[10)] $m_H \approx \sqrt{2}m_t$ which is also reported in the famous paper by Connes and Lott in Ref.1) where λ, given essentially by replacing $\mathrm{tr}(a^{(f)}a^{(f)\dagger})^2$ in Eq.(53) with $\mathrm{tr}(a^{(f)}a^{(f)\dagger})^2 - N_g^{-1}(\mathrm{tr}a^{(f)}a^{(f)\dagger})^2$, vanishes for $N_g = 1$. The appearance of the parameter ϵ completely makes the mass relation ambiguous, but we expect that ϵ^2 is of order unity.

§5 Matrix Method in the Chiral Space

This section is essentially a repetition of the previous section using the matrix notation in the chiral space:

$$\Psi(x) = \begin{pmatrix} \psi_L(x) = \psi(x,+) \\ \psi_R(x) = \psi(x,-) \end{pmatrix} \tag{57}$$

and

$$\mathcal{D}_\mu(x) = \begin{pmatrix} D_\mu^L(x) - \frac{1}{4}c_L\gamma_\mu & \frac{i}{4}\gamma_\mu H(x) \\ \frac{i}{4}\gamma_\mu H^\dagger(x) & D_\mu^R(x) - \frac{1}{4}c_R\gamma_\mu \end{pmatrix}, \tag{58}$$

[11] One can diagonalize all matrices $a^{(f)}, f = e, d, u$, to estimate Eq.(53) provided Kobayashi-Maskawa matrix appears for the charged current interactions.

where $D^L_\mu(x) = D_\mu(x,+)$, $D^R_\mu(x) = D_\mu(x,-)$, $c_L = C(+)$, $c_R = C(-)$, $H(x) = H(x,+)$ and $H^\dagger(x) = H(x,-)$.

The assignment (41) then leads to, omitting the generation and color indices,

$$\Psi = \begin{pmatrix} l_L \\ q_L \\ e_R \\ d_R \\ u_R \end{pmatrix}, \tag{59}$$

in terms of which the fermionic lagrangian (37) with Eqs.(38) and (39) is rewritten as

$$\mathcal{L}_D = i\bar{\Psi}\gamma^\mu \mathcal{D}_\mu \Psi, \tag{60}$$

where $\mathcal{D}_\mu$ is given by Eq.(58) with $D^{L,R}_\mu$ containing the unit matrix 1_{N_g}. We should insert the expressions for $D^L_\mu(x) = \partial_\mu + \mathcal{A}_\mu(x,+)$, $D^R_\mu(x) = \partial_\mu + \mathcal{A}_\mu(x,-), c_L = C(+), c_R = C(-)$ and $H(x) = H(x,+)$ from Eqs.(42), (43) and (44), respectively. The condition $C(y)M(y) = -M(y)C(-y)$ is translated into the one $c_L A + A c_R = A^\dagger c_L + c_R A^\dagger$, where $H(x,+)$ of Eq.(43) is rewritten as the product $H(x,+) = A\Phi_1(x)$ with $\Phi_1(x) = \begin{pmatrix} \phi(x) & 0 & 0 \\ 0 & \phi(x) & 0 \\ 0 & 0 & \tilde{\phi}(x) \end{pmatrix}$. It eliminates the linear terms in $c_{L,R}$ from the field strength[10)]

$$\mathcal{F}_{\mu\nu} = [\mathcal{D}_\mu, \mathcal{D}_\nu]. \tag{61}$$

For later purpose we introduce the associated field strength[12)]

$$\tilde{\mathcal{F}}_{\mu\nu} = \sum_\alpha h_\alpha^2 \Gamma_\alpha \mathcal{F}_{\mu\nu} \Gamma^\alpha, \tag{62}$$

where the sum over α runs over S, V, A, T and P corresponding to $\Gamma_\alpha = 1, \gamma_\lambda, \gamma_5\gamma_\lambda$, $\sigma_{\lambda\rho} = \frac{i}{2}[\gamma_\sigma, \gamma_\rho]$ and $i\gamma_5$, respectively. Putting

$$\begin{aligned} \textstyle\sum_\alpha h_\alpha^2 \Gamma_\alpha \gamma_\mu \Gamma^\alpha &= (h_S^2 - 2h_V^2 - 2h_A^2 + h_P^2)\gamma_\mu \equiv -\tfrac{2}{3}\alpha^2 \gamma_\mu, \\ \textstyle\sum_\alpha h_\alpha^2 \Gamma_\alpha \sigma_{\mu\nu} \Gamma^\alpha &= (h_S^2 - 4h_T^2 - h_P^2)\sigma_{\mu\nu} \equiv \tfrac{2}{3}\beta^2 \sigma_{\mu\nu}, \\ \textstyle\sum_\alpha h_\alpha^2 \Gamma_\alpha \Gamma^\alpha &= h_S^2 + 4h_V^2 - 4h_A^2 + 12h_T^2 - h_P^2 \equiv 1, \end{aligned} \tag{63}$$

we define the bosonic lagrangian

$$\mathcal{L}_B = -\frac{1}{32G^2}\mathrm{Tr}[C^2\gamma_0\mathcal{F}^{\dagger}_{\mu\nu}\gamma_0\tilde{\mathcal{F}}^{\mu\nu}], \tag{64}$$

where Tr means the trace over Dirac matrices, the 2×2 matrices in the chiral space and the internal symmetries and $C^2 = \begin{pmatrix} C_+^2 & 0 \\ 0 & C_-^2 \end{pmatrix}$. It is straightforward to show that Eq.(64) yields precisely the same lagrangian (52) for the same relations (51) with the parametrizations (53), (54), (55) (for $c_L = \varsigma 1_8$ and $c_R = -\varsigma 1_7$) and (56). In other words, the introduction of the associated field strength (62) with Eq.(63) reflects the independence of the inner products of the one-form χ and two-form $\chi\wedge\chi$ as exemplified in Eqs.(26) and (the last equation of) (35) with the same meaning of the parameters α^2 and β^2.

§6 Summary

To conclude we have been able to reconstruct the standard model within the framework of the modified formalism of the non-commutative differential geometry and to reinterpret the modified formalism in relation to Sogami's method[10)]. An obvious next question is how to extend the present formalism so as to describe more than one Higgs fields.

It is believed among NCG-minded people that NCG gives a constrained standard model[15)]. This conclusion depends on the choice of the starting, involutive algebra and Connes' definition of Yang-Mills-Higgs lagrangian through the Dixmier trace. In contrast, we reproduced the standard model without any constraints among the tree-level parameters. Our formalism parallels the ordinary differential geometry as closely as possible. Nonetheless, our reconstruction strongly depends on the pattern of existence of fermions.

The biggest departure from the ordinary differential geometry is the introduction of the extra one-form basis χ which does not vanish upon taking the wedge product and allows one to consider an algebraic sum of matrices of different types, namely, $A + B\chi$ (see below Eq.(12).) The latter aspect is only a convenient mathematical magic to treat gauge and Higgs fields in a unified way as a single, generalized one-form (16) where the shifted Higgs field appears. If we make use of the 2×2 matrix

rep[14)] considering the direct sum of Hilbert spaces $\mathcal{H}_+ \oplus \mathcal{H}_-$, χ twisted-commutes with bosonic and fermionic fields as shown in the Appendix A[12]. In this case, we may write Eq.(21) as

$$\mathcal{D}(x)\psi(x) = (D(x) + iC\chi + H(x)\chi)\psi(x), \tag{65}$$

where[13]

$$\begin{aligned} \psi(x) &= \begin{pmatrix} \psi(x,+) \\ \psi(x,-) \end{pmatrix}, \quad D(x) = \begin{pmatrix} D(x,+) & 0 \\ 0 & D(x,-) \end{pmatrix}, \\ C &= \begin{pmatrix} C(+) & 0 \\ 0 & C(-) \end{pmatrix}, \quad H(x) = \begin{pmatrix} 0 & H(x,+) \\ H(x,-) & 0 \end{pmatrix}. \end{aligned} \tag{66}$$

In this respect we recall that Sogami's generalized covariant derivative (58) is rewritten as[13)]

$$\mathcal{D}(x) \equiv \mathcal{D}_\mu(x) d\hat{x}^\mu = D(x) + iC\chi + \tilde{H}(x)\chi, \tag{67}$$

where

$$\chi = \frac{i}{4}\gamma_\mu d\hat{x}^\mu \tag{68}$$

acts on the spinor from the left and

$$D = \begin{pmatrix} D_\mu^L d\hat{x}^\mu & 0 \\ 0 & D_\mu^R d\hat{x}^\mu \end{pmatrix}, \quad C = \begin{pmatrix} c_L & 0 \\ 0 & c_R \end{pmatrix}, \quad \tilde{H} = \begin{pmatrix} 0 & H \\ H^\dagger & 0 \end{pmatrix}. \tag{69}$$

Therefore, Sogami's method[10)] provides us with a concrete realization of the mysterious symbol χ. It is important to remember, however, that our formalism does not presuppose a concrete realization of χ. Hence, even fermionic variable is multiplied by χ from both sides.

[12]Based on Z_2-graded algebra of Ref.14) we can assume that χ simply commutes with bosonic and fermionic variables.

[13]Cartan's structure equation of the connection one-form $\mathcal{A}(x)$ determines the curvature 2-form in this notation

$$\mathcal{F}(x) = d\mathcal{A}(x) + \mathcal{A}(x) \wedge \mathcal{A}(x),$$

where we put $\mathcal{D}(x) = d + \mathcal{A}(x)$ and adopt the convention[14)] of regarding $f = (f_1, f_2), f = D, C, H$, as elements of Z_2-graded algebra so that $f\chi = \chi f$. This is the matrix form of Eq.(28).

Last but not least we quote Ref.16) which precedes various works[2)−9)] on NCG approach to particle models. The authors in Ref.16) introduced the matrix derivation and proposed particle models which contain Higgs bosons belonging to the adjoint rep. Our allowance of taking an algebraic sum of matrices of different types, or equivalently, our introduction of the concept of Z_2-grading in the 2×2 matrix rep[14] fits to the fact that Higgs field in the theory belongs to any unitary rep as far as it couples to fermions.

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[14]The concept of Z_2-grading in the matrix formulation of NCG was first introduced in Ref.3) where the symbol χ was not yet introduced.

Appendix A

We shall add some comments on our interpretation of Eqs.(6) and (12).

As we showed in Ref.14), the relation (12) should be regarded as a calculational rule but not as a matrix equation. To see this let us employ the 2×2 matrix rep[14)]

$$
\begin{aligned}
f(x) &= \begin{pmatrix} f_1(x)=f(x,+) & 0 \\ 0 & f_2(x)=f(x,-) \end{pmatrix}, \quad \text{for even function } f(x,y), \\
g(x) &= \begin{pmatrix} 0 & g_1(x)=g(x,+) \\ g_2(x)=g(x,-) & 0 \end{pmatrix}, \quad \text{for odd function } g(x,y), \\
M &= \begin{pmatrix} 0 & M_1=M(+) \\ M_2=M(-) & 0 \end{pmatrix}, \quad \text{for odd function } M(y).
\end{aligned}
$$

Then Eq.(7) is brought to

$$
d_\chi f(x) = [Mf(x) - (-1)^{\partial f} f(x)M]\chi, \qquad (A\cdot 1)
$$

if we write

$$
\begin{aligned}
d_\chi f(x) &= d_\chi \begin{pmatrix} f_1(x) & 0 \\ 0 & f_2(x) \end{pmatrix} = \begin{pmatrix} 0 & d_\chi f_1(x) \\ d_\chi f_2(x) & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \partial_\chi f_1(x) \\ \partial_\chi f_2(x) & 0 \end{pmatrix} \chi = (\partial_\chi f(x))\chi, \\
d_\chi g(x) &= d_\chi \begin{pmatrix} 0 & g_1(x) \\ g_2(x) & 0 \end{pmatrix} = \begin{pmatrix} d_\chi g_1(x) & 0 \\ 0 & d_\chi g_2(x) \end{pmatrix} \\
&= \begin{pmatrix} \partial_\chi g_1(x) & 0 \\ 0 & \partial_\chi g_2(x) \end{pmatrix} \chi = (\partial_\chi g(x))\chi.
\end{aligned}
$$

Then Eq.(A· 1) makes sense as matrix equation yielding

$$
\begin{aligned}
d_\chi f_1(x) &= [M_1 f_2(x) - f_1(x) M_1]\chi, \quad d_\chi f_2(x) = [M_2 f_1(x) - f_2(x) M_2]\chi, \\
d_\chi g_1(x) &= [M_1 g_2(x) + g_1(x) M_2]\chi, \quad d_\chi g_2(x) = [M_2 g_1(x) + g_2(x) M_1]\chi.
\end{aligned}
$$

The reason for writing $d_\chi f$ and $d_\chi g$ in the above way lies in the fact that d_χ changes the grade. On the other hand, the Leibniz rule (11) written for ∂_χ is combined into

$$
\partial_\chi(f(x)g(x)) = (\partial_\chi f(x))g(x) + (-1)^{\partial f} f(x)\partial_\chi g(x). \qquad (A\cdot 2)
$$

The Leibniz rule (11) then reads

$$d_\chi(f(x)g(x)) = (d_\chi f(x))\tau_1 g(x)\tau_1 + (-1)^{\partial f} f(x) d_\chi g(x). \qquad (A\cdot 3)$$

Equation(A·2) implies Eq.(A·3) if

$$f(x)\chi = \chi\tau_1 f(x)\tau_1, \quad f(x): \text{ even or odd,} \qquad (A\cdot 4)$$

which is a matrix equation[15]. Here the matrix τ_1 plays a role of exchanging the indices $1\leftrightarrow 2$:

$$\begin{aligned}
\tau_1 \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix} \tau_1 &= \begin{pmatrix} f_2 & 0 \\ 0 & f_1 \end{pmatrix}, \\
\tau_1 \begin{pmatrix} 0 & g_1 \\ g_2 & 0 \end{pmatrix} \tau_1 &= \begin{pmatrix} 0 & g_2 \\ g_1 & 0 \end{pmatrix}.
\end{aligned}$$

We can recover (A·4) if the following relations are assumed to be valid:

$$f_1(x)\chi = \chi f_2(x), \quad f_2(x)\chi = \chi f_1(x), \quad f: \text{ even or odd,} \qquad (A\cdot 5)$$

which are nothing but Eq.(11). Equations (A·5) are not matrix equations.

The same is true also for Eq.(6). The reasoning is the same as above. Putting

$$\begin{aligned}
\psi(x) &= \begin{pmatrix} \psi_1(x) = \psi(x,+) \\ \psi_2(x) = \psi(x,-) \end{pmatrix} \\
C &= \begin{pmatrix} C_1 = C(+) & 0 \\ 0 & C_2 = C(-) \end{pmatrix}
\end{aligned}$$

we rewrite the third equation of Eq.(2) as

$$d_\chi \psi(x) = [M\psi(x) + iC\psi(x)]\chi,$$

where we should write

$$d_\chi \psi(x) = \begin{pmatrix} d_\chi \psi_1(x) \\ d_\chi \psi_2(x) \end{pmatrix} = \begin{pmatrix} \partial_\chi \psi_1(x) \\ \partial_\chi \psi_2(x) \end{pmatrix} \chi = (\partial_\chi \psi(x))\chi.$$

[15] A matrix equation involving χ should always take the form $A\chi = B\chi$, implying the usual matrix equation $A = B$. From Eq.(A·4) χ can not be a "scalar" in the matrix multiplication law. If, on the other hand, we regard $f = (f_1, f_2)$ as elements of a Z_2-graded algebra, we simply have[14)] $f\chi = \chi f$ which was employed in defining the curvature 2-form in the footnote on p.20.

Similarly, Eq.(4) written for the operator ∂_χ reads

$$\partial_\chi(f(x)\psi(x)) = (\partial_\chi f(x))\psi(x) + (-1)^{\partial f} f(x)\partial_\chi\psi(x). \qquad (A\cdot 6)$$

The Leibniz rule for the operator d_χ is obtained as

$$d_\chi(f(x)\psi(x)) = (d_\chi f(x))\tau_1\psi(x) + (-1)^{\partial f} f(x) d_\chi\psi(x). \qquad (A\cdot 7)$$

Hence, we have

$$\psi(x)\chi = \chi\tau_1\psi(x), \quad \tau_1 \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \psi_2 \\ \psi_1 \end{pmatrix}. \qquad (A\cdot 8)$$

This matrix equation is also interpreted as indicating a mere mathematical rule

$$\chi\psi_1(x) = \psi_2(x)\chi, \quad \chi\psi_2(x) = \psi_1(x)\chi, \qquad (A\cdot 9)$$

which is nothing but Eq.(6). This allows us to consistently apply the matrix multiplication rule in our formalism.

Appendix B

To compare with the formalism of Ref.11), we insert here an additional remark concerning the gauge status of Higgs field. To be more precise we shall rederive Eq.(18) in a different way more close to that of Ref.11).

Consider $\psi(x)$ in the Appendix A as a section of spinor bundle S. Locally it is expanded in terms of the basis $\{\mathbf{E}_K(x)\}$ of fibres $S_x = \{S_x^+, S_x^-\}$, to be called local frame fields:

$$\psi(x) = \sum_K \mathbf{E}_K(x)\psi^K(x) \equiv \begin{pmatrix} \sum_{k=1}^{k=m} \mathbf{e}_k(x,+)\psi_1^k(x) \\ \sum_{l=1}^{l=n} \mathbf{e}_l(x,-)\psi_2^l(x) \end{pmatrix}. \qquad (B\cdot 1)$$

The right-hand side may be calculated through the matrix multiplication rule by representing

$$\mathbf{E}_K(x) = \begin{pmatrix} \mathbf{e}_k(x,+) & 0 \\ 0 & \mathbf{e}_l(x,-) \end{pmatrix}$$

and performing the sum over the indices $\{k,l\}$ after multiplication with

$$\psi^K(x) = \begin{pmatrix} \psi_1^k(x) \\ \psi_2^l(x) \end{pmatrix}.$$

In what follows we follow this convention. The fibre spaces S_x^+ and S_x^- may have different dimensions, m and n, respectively, with corresponding basis $\{\mathbf{e}_k(x,+)\}_{k=1,\cdots,m}$ and $\{\mathbf{e}_l(x,-)\}_{l=1,\cdots,n}$. The index K takes on the values 1,2,$\cdots$, $(m+n)$. The covariant derivative in the discrete direction, denoted ∇_χ here, is defined by

$$\nabla_\chi\psi(x) = \sum_K[\nabla_\chi\mathbf{E}_K(x).\tau_1\psi^K(x) + \mathbf{E}_K(x)d_\chi\psi^K(x)], \qquad (B\cdot 2)$$

where we have used the Leibniz rule (A·7). Since $\nabla_\chi\mathbf{E}_K(x)$ can be written as a linear combination of the basis $\mathbf{E}_K(x)$, we put

$$\nabla_\chi\mathbf{E}_K(x) = \sum_L \mathbf{E}_L(x)\Phi^L_{\ K}(x)\chi \qquad (B\cdot 3)$$

where the connection form $\Phi(x) = (\Phi^L_{\ K}(x))$ comprises an odd function:

$$\Phi(x) = (\Phi^L_{\ K}(x)) = \begin{pmatrix} 0 & \Phi_1(x) \\ \Phi_2(x) & 0 \end{pmatrix}.$$

Thus we have

$$\begin{aligned}&\sum_K \nabla_\chi\mathbf{E}_K(x)\tau_1\psi^K(x) \\ &= \begin{pmatrix} \sum_{l=1}^{l=m}\mathbf{e}_l(x,+)\sum_{k=1}^{k=n}\Phi^l_{1\ k}(x)\psi^k_2(x) \\ \sum_{l=1}^{l=n}\mathbf{e}_l(x,-)\sum_{k=1}^{k=m}\Phi^l_{2\ k}(x)\psi^k_1(x)\end{pmatrix}\chi.\end{aligned}$$

The extra exterior derivative $d_\chi\psi^K(x)$ is given by

$$d_\chi\psi^K(x) = \sum_L(M^K_{\ L} + iC^K_{\ L})\psi^L(x)\chi$$

where the matrices $M = (M^K_{\ L})$ and $C = (C^K_{\ L})$ are the same ones as defined in the previous Appendix A so that

$$\begin{aligned}&\sum_K \mathbf{E}_K(x)d_\chi\psi^K(x) = \\ &\begin{pmatrix} \sum_{l=1}^{l=m}\mathbf{e}_l(x,+)\sum_{k=1}^{k=m}iC^l_{1\ k}(x)\psi^k_1(x) + \sum_{l=1}^{l=m}\mathbf{e}_l(x,+)\sum_{k=1}^{k=n}M^l_{1\ k}(x)\psi^k_2(x) \\ \sum_{l=1}^{l=n}\mathbf{e}_l(x,-)\sum_{k=1}^{k=m}M^l_{2\ k}(x)\psi^k_1(x) + \sum_{l=1}^{l=n}\mathbf{e}_l(x,-)\sum_{k=1}^{k=n}iC^l_{2\ k}(x)\psi^k_2(x)\end{pmatrix}\chi.\end{aligned}$$

Substituting them back into Eq.(B·2) we find that

$$\begin{aligned}\nabla_\chi\psi(x) &= \sum_{L,K}\mathbf{E}_L(x)H^L_{\ K}(x)\psi^K(x)\chi = (H(x)+iC)\psi(x)\chi \\ &\equiv \begin{pmatrix} \sum_{k=1}^{k=m}\mathbf{e}_k(x,+)(i\sum_{l=1}^{l=m}C^k_{1\ l}(x)\psi^l_1(x) + \sum_{l=1}^{l=n}H^k_{1\ l}(x)\psi^l_2(x)) \\ \sum_{l=1}^{l=n}\mathbf{e}_l(x,-)(i\sum_{l=1}^{l=n}C^k_{2\ l}(x)\psi^l_2(x) + \sum_{k=1}^{k=m}H^l_{2\ k}(x)\psi^k_1(x))\end{pmatrix}\chi,\end{aligned}$$

where we have put $H = \Phi + M$.

We finally determine the transformation property of the connection form $\Phi(x)$ under the transformation of the local frame fields:

$$
\begin{aligned}
{}^{g}\mathbf{E}_K(x) &= \sum_L \mathbf{E}_L(x) {(g^{-1})^L}_K(x) \\
&= \begin{pmatrix} \sum_{l=1}^{l=m} \mathbf{e}_l(x,+) {(g_1^{-l})^l}_k(x) & 0 \\ 0 & \sum_{l=1}^{l=n} \mathbf{e}_l(x,-) {(g_2^{-l})^l}_k(x) \end{pmatrix},
\end{aligned}
$$

where

$$
g(x) = ({g^L}_K(x)) = \begin{pmatrix} g_1(x) & 0 \\ 0 & g_2(x) \end{pmatrix}
$$

is the gauge transformation function[16]. Noting Eq.(A·3) it follows that

$$
\begin{aligned}
\nabla_\chi {}^{g}\mathbf{E}_K(x) &= \sum_L [\nabla_\chi \mathbf{E}_L(x).\tau_1 {(g^{-1})^L}_K(x)\tau_1 + \mathbf{E}_L(x).d_\chi {(g^{-1})^L}_K(x)] \\
&= \sum_{L,P} \mathbf{E}_L(x) {\Phi^L}_P(x)\chi\tau_1 {(g^{-1})^P}_K(x)\tau_1 + \sum_L \mathbf{E}_L(x).d_\chi {(g^{-1})^L}_K(x).
\end{aligned}
$$

Defining the transformed connection form by

$$
\nabla_\chi {}^{g}\mathbf{E}_K(x) = \sum_L {}^{g}\mathbf{E}_L(x) \, {}^{g}{\Phi^L}_K(x)\chi = \sum_{L,P} \mathbf{E}_L(x) {(g^{-1})^L}_P(x) \, {}^{g}{\Phi^P}_K(x)\chi
$$

and noting

$$
\sum_{L,P} \mathbf{E}_L(x) {\Phi^L}_P(x)\chi\tau_1 {(g^{-1})^P}_K(x)\tau_1 = \sum_{L,P} \mathbf{E}_L(x) {\Phi^L}_P(x) {(g^{-1})^P}_K(x)\chi
$$

$$
= \begin{pmatrix} 0 & \sum_{l=1}^{l=m} \mathbf{e}_l(x,+) \sum_{p=1}^{p=n} \Phi^l_{1p}(x) {(g_2{}^{-1})^p}_k(x) \\ \sum_{l=1}^{l=n} \mathbf{e}_l(x,-) \sum_{p=1}^{p=m} \Phi^l_{2p}(x) {(g_1{}^{-1})^p}_k(x) & 0 \end{pmatrix} \chi
$$

$$
\sum_{L,P} \mathbf{E}_L(x) {(g^{-1})^L}_P \, {}^{g}{\Phi^P}_K(x)
$$

$$
= \begin{pmatrix} 0 & \sum_{l=1}^{l=m} \mathbf{e}_l(x,+) \sum_{p=1}^{p=m} {(g_1^{-1})^l}_p(x) \, {}^{g}{\Phi_1{}^p}_k(x) \\ \sum_{l=1}^{l=n} \mathbf{e}_l(x,-) \sum_{p=1}^{p=n} {(g_2^{-1})^l}_p(x) \, {}^{g}{\Phi_2{}^p}_k(x) & 0 \end{pmatrix}
$$

[16]Strictly speaking, we should write $\rho(g(x))$ in place of $g(x)$. We can neglect the difference as far as Higgs field is concerned.

$$\sum_L \mathbf{E}_L(x)(d_\chi g^{-1})^L{}_K$$

$$= \begin{pmatrix} 0 & \sum_{l=1}^{l=m} \mathbf{e}_l(x,+)(d_\chi g_1^{-1})^l{}_k(x) \\ \sum_{l=1}^{l=n} \mathbf{e}_l(x,-)(d_\chi g_2^{-1})^l{}_k(x) & 0 \end{pmatrix},$$

we get

$$\begin{aligned} {}^g\Phi_1(x) &= g_1(x)\Phi_1(x)g_2^{-1} + g_1(x)\partial_\chi g_2(x), \\ {}^g\Phi_2(x) &= g_2(x)\Phi_1(x)g_1^{-1} + g_2(x)\partial_\chi g_1(x), \end{aligned}$$

where we have used Eq.(A·4) and the orthonormality:

$$\begin{aligned} \mathbf{e}_k(x,+)\cdot\mathbf{e}_l(x,+) &= \delta_{kl}, \quad k,l = 1,2,\cdots,m, \\ \mathbf{e}_k(x,-)\cdot\mathbf{e}_l(x,-) &= \delta_{kl}, \quad k,l = 1,2,\cdots,n. \end{aligned}$$

In the 2×2 matrix rep the result is converted into

$${}^g\Phi(x) = g(x)\Phi(x)g^{-1}(x) + g(x)\partial_\chi g^{-1}(x).$$

This is identical with Eq.(18) in the text. Note that, in general, $m \neq n$ in our formalism and the connection form $\Phi(x)$ of Eq.(B·3), or its *back-shifted* Higgs field $H(x)$ can never been related to the metric $\mathbf{E}_K(x)\cdot\mathbf{E}_L(x) = \delta_{KL}$. Consequently, the conclusion of Ref.11) concerning the unitarity of Higgs field is not applicable to the present formalism.

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